PMM U.S.S.R.,Vol.46,pp.205-209
0021-8928/83/2 0205 $7.50/0

UDC 539.3

SOURCES OF STRESS IN TWO HALF-SPACES*

V.P. KOCHUROV

Solutions are given for two versions of the problem of displacements from a source of stresses (the domain of displacement or stress field peculiarities) in two isotropic linearly elastic half-spaces, on whose plane boundaries conditions relating the boundary values of the derivatives of the displacements of different orders are satisfied. The method of solution proposed permits expressing the displacement in the half-spaces in terms of indefinite integrals of the gradients of the displacements produced by the stress source in an unbounded homogeneous medium.

1. Formulation of the problem. Two isotropic linearly-elastic half-spaces $z > h/2$ and $z < -h/2$ (z is the Cartesian coordinate and h is a constant) with shear moduli μ and μ_1, respectively, and transverse strain coefficients ν and ν_1 are bonded by means of a flat layer $-h/2 < z < h/2$ whose properties are given by the following relationship between the strain ε_2 and stress σ_2 tensors:

$$0.5\mu^{-1}\sigma_2 = \varkappa_2\mathbf{J}\cdot\varepsilon_2\cdot\mathbf{J} + \varkappa_3(\mathbf{kk}\cdot\varepsilon_2\cdot\mathbf{J} + \mathbf{J}\cdot\varepsilon_2\cdot\mathbf{kk}) + \varkappa_4\mathbf{kk}\cdot\varepsilon_2\cdot\mathbf{kk} + \varkappa_2\nu_2\zeta_2\mathbf{J}\cdot\cdot\varepsilon_2\mathbf{J} \quad (-h/2 < z < h/2) \tag{1.1}$$

Here $\mathbf{k}$ is the unit vector normal to the middle plane of the layer $z = 0$, directed toward the halfspace $z > h/2$, $\mathbf{J} \equiv \mathbf{I} - \mathbf{kk}$, $\mathbf{I}$ is a unit tensor of the second rank, $\varkappa_2, \varkappa_3, \varkappa_4, \nu_2$ are dimensionless positive constants, $\zeta_i \equiv 1/(1-2\nu_i)$ $(i = 0, 1, 2)$; the point denote the scalar tensor convolution operator (a vector is considered a first rank tensor); two vectors between which there are no addition, subtraction, multiplication, and equality signs form a dyad; quantities (constants and functions) referring to the half-space $z > h/2$ are provided with the subscript 0 which is ordinarily omitted (only $\mathbf{u}$ and $\mathbf{u}_0$ should be distinguished, where their interrelationship is described by (1.2) below): quantities referring to the half-space $z < -h/2$ are marked with the subscript 1 (the exception is $\varkappa \equiv \mu_1/\mu$) and quantities referring to the plane layer, by the subscripts 2,3,4.

To particular cases (called "problem 1" and "problem 2") of the following problem are considered: Find the displacements $\mathbf{u}$ and $\mathbf{u}_1$ and the stresses σ and σ_1, respectively, in the half-spaces $z > h/2$ and $z < -h/2$ due to a source of stress (zero-dimensional, one-dimensional, two-dimensional, or three-dimensional domain of singularities of the displacement or stress fields) of finite sizes in the half-space $z > h/2$.

The displacement $\mathbf{u}_\infty$ and stress σ_∞ fields produced by the source of stress in an unlimited homogeneous medium with the elastic constants μ and ν are considered known. On the functions $\mathbf{u}_\infty$, $\mathbf{u}_1$ and $\mathbf{u}_0$, where

$$\mathbf{u}_0 = \mathbf{u} - \mathbf{u}_\infty \quad (z > h/2) \tag{1.2}$$

there are imposed regularity conditions at infinity

$$|\nabla^\alpha \mathbf{u}_i| < \frac{M}{R^\alpha}(R\to\infty)\begin{pmatrix} i=0, \alpha=0,\ 1 \ \text{as} \ z > h/2 \\ i=1, \alpha=0,\ 1 \ \text{as} \ z < -h/2 \\ i=2, \alpha=0,\ 1 \ \text{as} \ -h/2 < z < h/2 \\ i=\infty, \alpha=0,1,2,3 \ \text{as} \ -\infty < z < \infty \end{pmatrix} \tag{1.3}$$

where $\nabla \equiv \partial/\partial\mathbf{r}$, $\mathbf{r}$ is the radius-vector of the point of observation, R is the distance to the source, M is a positive constant $(M < \infty)$, $\nabla^\alpha \equiv \nabla\nabla\ldots\nabla$, α times (polyadic product). The stress tensor corresponding to the displacements $\mathbf{u}_0$ in the medium with elastic constants μ and ν will be denoted by σ_0.

The stress sources considered in the literature /1-3/ are a point force and a combination of forces, Somigliana dislocations, centers of expansion, etc. and satisfy the condition of decreasing displacements and their derivatives at infinity (1.3) (for $i = \infty$).

Additional assumptions are taken in the following two modifications.

Problem 1

$$\nu_2 = 0,\quad \varkappa_4 = \infty,\quad \varkappa_3 = o(1),\quad h = o(z'),\quad \varkappa_2 h = O(z'),\quad h/\varkappa_3 = O(z')$$

*Prikl.Matem.Mekhan.,46,No.2,pp.272-277,1982

Problem 2

$$\varkappa_4 = \infty, \quad h = o(z'), \quad \varkappa_2 h = O(z'), \quad \varkappa_3 h = O(z')$$

Here z' is the distance between the stress source and the middle surface of the layer $z = 0$.

The boundary value problem for three phases is reduced to a problem for two half-spaces. To this end, the method of exclusion of a thin film, described in /4,5/ in application to plane problems of elasticity theory, can be utilized. In /4/ and in the equations of the theory of thin-walled shells relationships are derived between the stresses and displacements on the boundaries of two half-planes separated by a thin curvilinear film. In /5/, approximate conditions on the half-plane boundary are found in an investigation of the interaction between an edge dislocation and a thin film on the rectilinear edge of a half-plane, from the equilibrium equations and the governing equations of the film material by expanding the displacements in the latter in a power series in the distance from the phase separation line.

In the case of a spatial state of stress under consideration here, the method of expanding the displacement in a plane layer in a series permits finding the solution of the first boundary value problem for a layer (with displacements given on the boundary) to any required accuracy. In particular, if terms containing the parameter h in powers higher than the first are discarded in series describing the stress and displacement vectors on the interphasal boundaries, then we obtain the following conditions on the half-space boundaries

$$\begin{gathered}\lim_{z\to h/2+0} \{h\nabla\cdot\mathbf{J}\cdot[\varkappa_2(\nabla\mathbf{u}+\zeta_2\mathbf{u}\nabla)\cdot\mathbf{J}+a\nabla\mathbf{u}\cdot\mathbf{kk}]+2\mu^{-1}\mathbf{k}\cdot\boldsymbol{\sigma}\}+\\ \lim_{z\to -h/2-0} \{h\nabla\cdot\mathbf{J}\cdot[\varkappa_2(\nabla\mathbf{u}_1+\zeta_2\mathbf{u}_1\nabla)\cdot\mathbf{J}+a\nabla\mathbf{u}_1\cdot\mathbf{kk}]-2\mu^{-1}\mathbf{k}\cdot\boldsymbol{\sigma}_1\}=0\\ \lim_{z\to h/2+0}(bh\mu^{-1}\mathbf{k}\cdot\boldsymbol{\sigma}\cdot\mathbf{J}-\mathbf{u})+\lim_{z\to -h/2-0}(bh\mu^{-1}\mathbf{k}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{J}+\mathbf{u}_1)=0\end{gathered} \tag{1.4}$$

Here in problem 1

$$a = 0, \quad b = 0.5/\varkappa_3, \quad \zeta_2 = 1 \tag{1.5}$$

in problem 2

$$a = \varkappa_3, \quad b = 0 \tag{1.6}$$

The boundary (1.4), (1.5) combines the properties of a membrane and a layer that is elastically resistive to the mutual slip of the half-spaces. It is an extension of the cohesion and slip boundaries known in the /6,7/. The boundary (1.4), (1.6) is imparted with the properties of a membrane plate (is elastically resistive to strains in the intrinsic middle plane and to shears in the transverse direction).

The problems formulated are distinct from the problem examined in /8/ in that the boundary conditions (1.4) interrelate boundary values of derivatives of different orders for the displacements in the half-spaces.

2. Solution of problem 1. Instead of the fields $\mathbf{u}$ and $\boldsymbol{\sigma}$ of the half-space $z > h/2$ we seek the fields $\mathbf{u}_{0*}$ and $\boldsymbol{\sigma}_{0*}$, mirror images of the fields $\mathbf{u}_0$ and $\boldsymbol{\sigma}_0$, respectively, with respect to the $z = 0$ plane. The functions $\mathbf{u}_{0*}$ and $\boldsymbol{\sigma}_{0*}$ satisfy the elasticity theory equations (with the elasticity constants μ and ν) in the half-space $z < -h/2$, and the regularity conditions at infinity. Taking account of the field reflection operations the boundary conditions (1.4) and (1.5) can be formulated as follows:

$$\begin{gathered}\lim_{z\to -h/2-0} \{\varkappa_2 h\nabla\cdot\mathbf{J}\cdot[\nabla(\mathbf{u}_{0*}\cdot\mathbf{A}+\mathbf{u}_-+\mathbf{u}_1)+(\mathbf{A}\cdot\mathbf{u}_{0*}+\mathbf{u}_-+\mathbf{u}_1)\nabla]\cdot\mathbf{J}+\\ 2\mu^{-1}k\cdot(-\boldsymbol{\sigma}_{0*}\cdot\mathbf{A}+\boldsymbol{\sigma}_--\boldsymbol{\sigma}_1)\}=0\\ \lim_{z\to -h/2-0}[h\mu^{-1}\mathbf{k}\cdot(-\boldsymbol{\sigma}_{0*}\cdot\mathbf{A}+\boldsymbol{\sigma}_-+\boldsymbol{\sigma}_1)\cdot\mathbf{J}-2\varkappa_3(\mathbf{A}\cdot\mathbf{u}_{0*}+\mathbf{u}_--\mathbf{u}_1)]=0\end{gathered} \tag{2.1}$$

Here $\mathbf{A} \equiv \mathbf{I} - 2\mathbf{kk}$ is the reflection tensor, $\mathbf{u}_-$ and $\boldsymbol{\sigma}_-$ are fields obtained by displacing the fields corresponding to $\mathbf{u}_\infty$ and $\boldsymbol{\sigma}_\infty$ $(z > h/2)$ by the thickness of the plane layer h towards negative values of z. A representation in terms of vector harmonic functions $\boldsymbol{\omega}_{0*}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_-$ in the half-space $z < -h/2$ is used for the dispalcements $\mathbf{u}_{0*}, \mathbf{u}_1, \mathbf{u}_-$

$$\begin{gathered}\mathbf{u}_i = \mathbf{k}\cdot\nabla\boldsymbol{\omega}_i - \alpha_i(z+h/2)\nabla^2\cdot\boldsymbol{\omega}_i \quad (z < -h/2)\\ (\alpha_{0*} = \alpha_- = \alpha \equiv 1/(3-4\nu), \ \alpha_1 \equiv 1/(3-4\nu_1))\end{gathered} \tag{2.2}$$

Formulas (2.2) express the solutions of elasticity theory problems for a half-space with displacements given on the boundary /3/

$$\boldsymbol{\omega}_i = \frac{1}{2\pi}\int_{\Gamma_1}\frac{\mathbf{U}_i\, d\Gamma_1}{R_1} \quad (z < -h/2), \quad \mathbf{U}_i = \lim_{z\to -h/2}\mathbf{u}_i \ (i = 0*, 1, -) \tag{2.3}$$

(R_1 is the distance to the element, $d\Gamma_1$ of the interfacial surface, Γ_1 between the plane layer and the half-space $z < -h/2$). With respect to the function $\mathbf{u}_-$, the representation (2.2) ($i = -$) can be considered still as the analytic continuation of values of the function u_- in the plane $z = -h/2$ to the half-space $z < -h/2$.

Substitution of (2.2) into the boundary conditions (2.1) results in the following boundary conditions for the harmonic functions

$$\lim_{z\to -h/2-0} \{\varkappa_2 h (\mathbf{k}\cdot\nabla)\mathbf{J}\cdot[-\nabla^2 + (\mathbf{k}\cdot\nabla)^2\mathbf{I}]\cdot\mathbf{J}\cdot(\mathbf{A}\cdot\boldsymbol{\omega}_{0*} + \boldsymbol{\omega}_- + \boldsymbol{\omega}_1) + 2[-\alpha\nabla^2 - (1-2\alpha)(\mathbf{k}\cdot\nabla)\nabla\mathbf{k} + (\mathbf{k}\cdot\nabla)\mathbf{k}\nabla + (\mathbf{k}\cdot\nabla)^2\mathbf{I}]\cdot(\mathbf{A}\cdot\boldsymbol{\omega}_{0*} + \boldsymbol{\omega}_-) + 2\varkappa[-\alpha_1\nabla^2 + (\mathbf{k}\cdot\nabla)\nabla\mathbf{k} - (1-2\alpha_1)(\mathbf{k}\cdot\nabla)\mathbf{k}\nabla + (\mathbf{k}\cdot\nabla)^2\mathbf{I}]\cdot\boldsymbol{\omega}_1\} = 4\lim_{z\to -h/2-0}\boldsymbol{\Upsilon} \tag{2.4}$$

$$\lim_{z\to -h/2-0}[h\mathbf{J}\cdot\{[-\alpha\nabla^2 - (1-2\alpha)(\mathbf{k}\cdot\nabla)\nabla\mathbf{k} + (\mathbf{k}\cdot\nabla)^2\mathbf{I}]\cdot(\mathbf{A}\cdot\boldsymbol{\omega}_{0*} + \boldsymbol{\omega}_-) - \varkappa[-\alpha_1\nabla^2 + (\mathbf{k}\cdot\nabla)\nabla\mathbf{k} + (\mathbf{k}\cdot\nabla)^2\mathbf{I}]\cdot\boldsymbol{\omega}_1\} + 2\varkappa_3(\mathbf{k}\cdot\nabla)(\mathbf{A}\cdot\boldsymbol{\omega}_{0*} + \boldsymbol{\omega}_- - \boldsymbol{\omega}_1)] = 2h\lim_{z\to -h/2-0}\mathbf{J}\cdot\boldsymbol{\Upsilon}$$

$$\boldsymbol{\Upsilon} \equiv [-\alpha\nabla^2 + \alpha(\mathbf{k}\cdot\nabla)(\nabla\mathbf{k} + \mathbf{k}\nabla) + (\mathbf{k}\cdot\nabla)^2\mathbf{I}]\cdot\boldsymbol{\omega}_- \quad (z < -h/2) \tag{2.5}$$

Operator expressions are used in (2.4) and (2.5) and henceforth, in which it is understood that the Hamilton operator ∇ acts on the function of $\mathbf{r}$ nearest on its "right"; by definition this operator does not act on a function of z, hence $\nabla(z\mathbf{T}) \equiv z\nabla\mathbf{T} \equiv z(\nabla\mathbf{T})$ ($\mathbf{T}$ is an arbitrary tensor function of $\mathbf{r}$); the gradient of z is always denoted by $\mathbf{k}$.

The left and right sides in conditions (2.4) are the boundary values of certain harmonic functions expressed in terms of $\boldsymbol{\Upsilon}$ and the partial derivatives of $\boldsymbol{\omega}_{0*}$, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_-$. According to the Dirichlet theorem /3/, the functions mentioned should agree in the whole half-space $z < -h/2$. Therefore, we obtain a system of two vector partial differential equations of the form (2.4) with the limit signs removed (lim as $z\to -h/2-0$) for the harmonic functions $\boldsymbol{\omega}_{0*}$ and $\boldsymbol{\omega}_1$ in the domain $z < -h/2$.

The solution of the system of equations for the functions $\boldsymbol{\omega}_{0*}$ and $\boldsymbol{\omega}_1$ is found in the form

$$\boldsymbol{\omega}_i = A_i\int_{-\infty}^{z}dz\int_{-\infty}^{z}\nabla^2\cdot\boldsymbol{\omega}_- dz + B_i\int_{-\infty}^{z}\nabla\mathbf{k}\cdot\boldsymbol{\omega}_- dz + C_i\int_{-\infty}^{z}\mathbf{k}\nabla\cdot\boldsymbol{\omega}_- dz + D_i\mathbf{k}\mathbf{k}\cdot\boldsymbol{\omega}_i + E_i\boldsymbol{\omega}_- + \frac{1}{h}\sum_{j=1}^{K}\exp\left(-Q_j\frac{z}{h}\right)\int_{-\infty}^{z}\exp\left(Q_j\frac{z}{h}\right)\times \left(A_{ij}\int_{-\infty}^{z}dz\int_{-\infty}^{z}\nabla^2\cdot\boldsymbol{\omega}_- dz + B_{ij}\int_{-\infty}^{z}\nabla\mathbf{k}\cdot\boldsymbol{\omega}_- dz + C_{ij}\int_{-\infty}^{z}\mathbf{k}\nabla\cdot\boldsymbol{\omega}_- dz + D_{ij}\mathbf{k}\mathbf{k}\cdot\boldsymbol{\omega}_- + E_{ij}\boldsymbol{\omega}_-\right)dz \quad (z < -h/2;\ i = 0*, 1) \tag{2.6}$$

where

$$Q_j,\ A_i,\ B_i,\ C_i,\ D_i,\ E_i,\ A_{ij},\ B_{ij},\ C_{ij},\ D_{ij},\ E_{ij},\ (i = 0*,\ 1;\ j = 1, 2, \ldots, K)$$

are constants (complex numbers), K is a positive integer. Such a form of the solution is due to the property of harmonicity of the required functions $\boldsymbol{\omega}_{0*}$, $\boldsymbol{\omega}_1$ and the known function $\boldsymbol{\omega}_-$. Upon substituting (2.6) into the vector equations, a system of 54 ($K = 4$) linear algebraic equations is obtained, from which all the constants of the right side of (2.6) the eigennumbers Q_j ($j = 1, 2, 3, 4$) are nontrivial solutions of subsystems with zero determinants are found (expressed in terms of the elastic constants of the materials). After this, by using the representations (2.2) ($i = 0*, 1$), and

$$\boldsymbol{\omega}_- = \int_{-\infty}^{z}\mathbf{u}_- dz + 0.5\zeta\int_{-\infty}^{z}(z + h/2)\,dz\int_{-\infty}^{z}\nabla^2\cdot\mathbf{u}_- dz \quad (z < -h/2) \tag{2.7}$$

in formulas (2.6), the passage from the functions $\boldsymbol{\omega}_{0*}$ $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_-$ to the functions $\mathbf{u}_{0*}$, $\mathbf{u}_1$, $\mathbf{u}_-$ was accomplished and also the necessary field reflection operations were performed. In final form, the displacements in the half-spaces $\mathbf{u}$ and $\mathbf{u}_1$ were expressed by indefinite integrals of partial derivatives of the functions $\mathbf{u}_*$ and $\mathbf{u}_-$, where $\mathbf{u}_*$ is a vector that is the mirror image of the vector $\mathbf{u}_\infty$ ($z < h/2$) with respect to the plane $z = h/2$.

3. Particular cases of problem 1. For $h = 0$ ($\varkappa_2 < \infty$; $\varkappa_3 > 0$) the case of two bonded half-spaces holds:

$$\lim_{z\to+0}\mathbf{u} = \lim_{z\to-0}\mathbf{u}_1, \qquad \lim_{z\to+0}\mathbf{k}\cdot\boldsymbol{\sigma} = \lim_{z\to-0}\mathbf{k}\cdot\boldsymbol{\sigma}_1 \tag{3.1}$$

The solution of this problem is obtained by expanding the solution of problem 1 in a power series in small parameters and discarding terms containing such parameters:

$$\mathbf{u} = \mathbf{u}_\infty + \frac{1-\varkappa}{\gamma}\mathbf{u}_* + \frac{\zeta}{\eta}\left(\frac{2}{\gamma} - \frac{1+2\alpha}{2\beta} - \frac{1}{2\beta_1}\right)\int\limits_\infty^z dz \int\limits_\infty^z \nabla^2\cdot\mathbf{u}_* dz + \tag{3.2}$$
$$(1-\varkappa)\left(\frac{1}{\beta} - \frac{1}{\gamma}\right)\int\limits_\infty^z \left(\nabla\mathbf{k} + \frac{\zeta}{\eta}\mathbf{k}\nabla\right)\cdot\mathbf{u}_* dz + \frac{\varkappa-1}{\beta}[2\mathbf{kk} +$$
$$2\alpha z(\mathbf{k}\nabla - \nabla\mathbf{k}) - \alpha\zeta z^2\nabla^2]\cdot\mathbf{u}_* \quad (z > 0)$$
$$\mathbf{u}_1 = \frac{2}{\gamma}\mathbf{u}_\infty + \frac{\zeta}{\eta}\left(\frac{2}{\gamma} - \frac{1+2\alpha}{2\beta} - \frac{1}{2\beta_1}\right)\int\limits_{-\infty}^z dz \int\limits_{-\infty}^z \nabla^2\cdot\mathbf{u}_\infty dz +$$
$$\frac{\zeta}{\eta}\left(\frac{\alpha}{\beta} - \frac{\alpha_1}{\beta_1}\right) z \int\limits_{-\infty}^z \nabla^2\cdot\mathbf{u}_\infty dz + \frac{\zeta}{\eta}\left(\frac{\alpha}{\beta} + \frac{1}{\beta_1} -\right.$$
$$\left.\frac{2}{\gamma}\right)\int\limits_{-\infty}^z \mathbf{k}\nabla\cdot\mathbf{u}_\infty dz + (1-\varkappa)\left(\frac{1}{\beta} - \frac{1}{\gamma}\right)\int\limits_{-\infty}^z \nabla\mathbf{k}\cdot\mathbf{u}_\infty dz \quad (z < 0)$$
$$\beta \equiv \alpha + \varkappa, \quad \beta_1 \equiv 1 + \alpha_1\varkappa, \quad \gamma \equiv 1 + \varkappa, \quad \eta \equiv 0{,}5/(1-\nu)$$

The problem (3.1) is solved by using the Trefftz representation in the paper /8/. In the particular case of a point force, the solution (3.2) agrees with the solution presented in /6/.

For $\varkappa_2 = 0$, $\varkappa_3 \to 0$, $h \to 0$, $\varkappa_3/h = o(1/z')$ the slip boundary conditions are obtained

$$\lim_{z\to+0}\mathbf{k}\cdot\mathbf{u} = \lim_{z\to-0}\mathbf{k}\cdot\mathbf{u}_1,\ \lim_{z\to+0}\mathbf{kk}\cdot\cdot\boldsymbol{\sigma} = \lim_{z\to-0}\mathbf{kk}\cdot\cdot\boldsymbol{\sigma}_1,\ \lim_{z\to+0}\mathbf{k}\cdot\boldsymbol{\sigma}\cdot\mathbf{J} = \lim_{z\to-0}\mathbf{k}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{J} = 0 \tag{3.3}$$

The solution of this problem has the following form

$$\mathbf{u} = \mathbf{u}_\infty + \mathbf{u}_* - \frac{1+\alpha_1}{2(\alpha_1\beta + \alpha\beta_1)}\left\{(1+\alpha)\int\limits_\infty^z dz \int\limits_\infty^z \nabla^2\cdot\mathbf{u}_* dz -\right. \tag{3.4}$$
$$2\int\limits_\infty^z [(1-\alpha)\nabla\mathbf{k} + (1+\alpha)\mathbf{k}\nabla]\cdot\mathbf{u}_* dz + 2[2\mathbf{kk} + 2\alpha z(\mathbf{k}\nabla - \nabla\mathbf{k}) -$$
$$\left.\alpha\zeta z^2\nabla^2]\cdot\mathbf{u}_*\right\} \quad (z > 0)$$
$$\mathbf{u}_1 = \frac{1+\alpha}{2(\alpha_1\beta + \alpha\beta_1)}\left\{\frac{\alpha_1\zeta}{\alpha\zeta_1}(1+\alpha)\int\limits_{-\infty}^z dz \int\limits_{-\infty}^z \nabla^2\cdot\mathbf{u}_\infty dz -\right.$$
$$2\int\limits_{-\infty}^z \left[(1-\alpha_1)\nabla\mathbf{k} + \left(\alpha_1\frac{\zeta}{\zeta_1} + 1\right)\mathbf{k}\nabla\right]\cdot\mathbf{u}_\infty dz +$$
$$\zeta\left(\frac{\alpha_1}{\alpha} - 1\right) z \int\limits_{-\infty}^z \nabla^2\cdot\mathbf{u}_\infty dz + 2\left[2\mathbf{kk} + 2\alpha_1\left(\frac{\zeta}{\zeta_1}\mathbf{k}\nabla - \nabla\mathbf{k}\right) -\right.$$
$$\left.\left.\alpha_1\zeta z^2\nabla^2\right]\cdot\mathbf{u}_\infty\right\} \quad (z < 0)$$

For a point force, a solution agreeing with the solution in /7/ is obtained from (3.4).

4. Solution of the problem 2. Problem 2 is solved by the method elucidated in Sect. 2. The vector harmonic functions $\boldsymbol{\omega}_{0*}$ and $\boldsymbol{\omega}_1$, giving the solution of this problem in conformity with (2.2) and (2.7), are expressed as follows:

$$\boldsymbol{\omega}_1 = \frac{1+\alpha}{\varkappa_2(1+\zeta_2)h}\int\limits_{-\infty}^z \exp\left(-p\,\frac{z-z_1}{h}\right)\left\{\frac{\beta+\beta_1}{q}\left[\frac{1}{\varkappa_2(1+\zeta_2)} -\right.\right. \tag{4.1}$$
$$\left.\left.\frac{1}{\varkappa_3}\right]\operatorname{sh}\left(q\,\frac{z-z_1}{h}\right) - 2\operatorname{ch}\left(q\,\frac{z-z_1}{h}\right)\right\}\left[\int\limits_{-\infty}^{z_1} dz_2 \int\limits_{-\infty}^{z_2} \nabla_3{}^2\cdot\boldsymbol{\omega}_-(\mathbf{r}_3)\,dz_3 -\right.$$
$$\left.\int\limits_{-\infty}^{z_1} (\nabla_2\mathbf{k} + \nabla_2\mathbf{k})\cdot\boldsymbol{\omega}_-(\mathbf{r}_2)\,dz_2 + \mathbf{kk}\cdot\boldsymbol{\omega}_-(\mathbf{r}_1)\right] dz_1 +$$
$$\frac{1+\alpha}{\varkappa_3 h}\int\limits_{-\infty}^z \exp\left(-p\,\frac{z-z_1}{h}\right)\left\{\frac{1}{q}\operatorname{sh}\left(q\,\frac{z-z_1}{h}\right)\left[(\beta+\beta_1)\left(\frac{1}{\varkappa_2(1+\zeta_2)} -\right.\right.\right.$$

$$\left.\frac{1}{\varkappa_3}\right)\mathbf{kk}\cdot\boldsymbol{\omega}_-(\mathbf{r}_1)+\frac{2(\beta_1-\beta)}{\varkappa_2(1+\zeta_2)}\int_{-\infty}^{z_1}(\nabla_2\mathbf{k}-\mathbf{k}\nabla_2)\cdot\boldsymbol{\omega}_-(\mathbf{r}_2)\,dz_2\Big]+$$

$$2\,\mathrm{ch}\left(q\frac{z-z_1}{h}\right)\mathbf{kk}\cdot\boldsymbol{\omega}_-(\mathbf{r}_1)\Big\}\,dz_1+\frac{2}{\varkappa_2 h}\int_{-\infty}^{z}\exp\left(-\frac{\gamma}{\varkappa_2}\frac{z-z_1}{h}\right)\times$$

$$\Big[\int_{-\infty}^{z_1}dz_2\int_{-\infty}^{z_2}\nabla_3{}^2\cdot\boldsymbol{\omega}_-(\mathbf{r}_3)\,dz_3-\int_{-\infty}^{z_1}(\nabla_2\mathbf{k}+\mathbf{k}\nabla_2)\cdot\boldsymbol{\omega}_-(\mathbf{r}_2)\,dz_2+$$

$$\boldsymbol{\omega}_-(\mathbf{r}_1)\Big]\,dz_1\;(z<-h/2)$$

$$\boldsymbol{\omega}_{0*}=\mathbf{A}\cdot(\boldsymbol{\omega}_1-\boldsymbol{\omega}_-)\;(z<-h/2),\;\nabla_n=\frac{\partial}{\partial\mathbf{r}_n}\;(n=1,2,3),$$

$$p\equiv\frac{\beta+\beta_1}{2}\left[\frac{1}{\varkappa_2(1+\zeta_2)}+\frac{1}{\varkappa_2}\right]$$

$$q\equiv\left[p^2-\frac{4\beta\beta_1}{\varkappa_2\varkappa_3(1+\zeta_2)}\right]^{1/2}\quad(q^2\geqslant 0)$$

Here z_1, z_2, z_3 are variables of integration, $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are obtained from $\mathbf{r}$ by replacing z by z_1, z_2, z_3, respectively. The triple integrals expressing the displacements in the half-spaces can be converted into single integrals by integration by parts. However, such a conversion results in more awkward depiction of the results, which becomes especially inconvenient in going over to the displacements from specific sources of stress.

The solution (3.2) is obtained in the particular case $h = 0$.

5. Two-dimensional boundary value problems. The solutions obtained for problems 1 and 2 are valid even for the plane state of strain if R in the regularity conditions at infinity (1.3) is understood to be the distance to the source on the plane. It should be kept in mind that the most important sources of stress, the point force /9/ and the dislocation /10/, do not satisfy the condition (1.3) (for $i=\infty$). Nevertheless, the uniqueness of the solution of the problem will be assured if each such source is considered as a component of a system of sources for which the sum of the characteristic vectors (the force vectors, the Burgers vectors, etc.) is zero.

REFERENCES

1. ESHELBY J., Continuum Theory of Dislocations /Russian translation/, Izd. Inostr. Lit., Moscow, 1963.
2. KOEHLER J.S., Elastic centers of strain and dislocations, J. Appl. Phys., Vol.37, No.12, 1966.
3. BLOKH V.I., Elasticity Theory. Izd.Khar'kovsk. Gos. Un-ta, 1964.
4. CHOBANIAN K.S. and KHACHIKIAN A.S., Plane state of strain of an elastic body with a thin-walled flexible inclusion, Izv. Akad. Nauk ArmSSR, Mekhanika, Vol.20, No.6, 1967.
5. CONNERS G.H., The interaction of a dislocation with a coated plane boundary. Intern. J. Engng. Sci., Vol.5, No.1, 1967.
6. RONGVED L., Force interior to one of two joined semi-infinite solids. In: Proc. Second Mid-west. Conf. on Solid Mechanics, 1955. Purdue Univ., Lafayette, 1955.
7. DUNDURS J. and HETÉNYI M., Transmission of force between two semi-infinite bodies. Trans. ASME, Ser. E. J. Appl. Mech., Vol.32, No.3, 1965.
8. MAKSIMOVICH V.N. and PLIATSKO G.V., Application of the method of inversion to determine the stresses in inhomogeneous bodies, Prikl. Mekhan., Vol.10, No.10, 1974.
9. MUSKHELISHVILI N.I., Some Basic Problems of Mathematical Theory of Elasticity. Groningen, Noordhoff, 1953.
10. De WIT R., Some relations for straight dislocations. Phys. Status Solidi, Vol.20, No.2, 1967.

Translated by M.D.F.